LAWS OF MOTION OF AN EARTH - SATELLITE

by

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Translated by

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DEFENCE RESEARCH BOARD

CORRECTION NOTE. DRB T 8[illegible] R.

Page 2, sixth line in second-last paragraph, read: descend into the denser layers of the atmosphere".

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LAWS OF MOTION OF AN EARTH-SATELLITE

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The appearance, in our natural environment, of artificial heavenly bodies, created by the hands of the Soviet people, has of course led many people to ask: How are they launched and why do they stay in flight?

While making no claim to describe precisely all the forces acting on a satellite, with due allowance for the asphericity of the earth and the resistance of the terrestrial atmosphere, we shall here attempt to give just the most general idea of the fundamental relationships and laws governing the launching of a satellite by a multi-stage rocket and its motion around the globe.

If in the first approximation we disregard the influence of the moon, of the other planets, and of the sun on the satellite's trajectory, we may draw ourselves the following picture of the satellite's motion in the spherical gravitational field of the earth.

For an artificial satellite to travel around the earth in a circular orbit and not fall to the surface, it is necessary, on the basis of D'Alembert's principle, that the weight of the satellite at any given height of the orbit should be equal to the centrifugal inertial force. Neglecting the attractions of the sun and planets, atmospheric resistance, and the non-central potential of the terrestrial gravitation, this condition may be expressed mathematically as:

$$mg = \frac{mV_{cr}^2}{r}$$

where m is the mass of the satellite; V_{cr} is its "circular" velocity, directed along the tangent to the orbit; r_o is the radius of the earth; r is the radius of the satellite orbit; g and g_o are the acceleration of gravity at height $h = r - r_o$ and at the earth's surface respectively.

From this equation it is not difficult to determine the velocity V_{cr} which the satellite must have in order to rotate around the earth at height h:

$$V_{cr} = \sqrt{gr}. \qquad (1)$$

Substituting in this expression the quantity

$$g = g_o(r_o:r)^2,$$

we finally obtain, since the acceleration of gravity decreases in inverse proportion to the square of the distance from the center of the earth, the expression:

$$V_{cr} = (V_{cr})_o \sqrt{\frac{r_o}{r}}. \qquad (2)$$

In the general case of a satellite traveling around the earth in an elliptical path, its velocity at any point of this path will be defined by the equation:

$$V_{el} = V_{cr}\sqrt{2 - \frac{r}{a}},$$

where a is the major semi-axis of the ellipse.

Consequently, the higher above the surface of the earth the path of the satellite lies, the less its "circular" velocity V_{cr} will need to be. The minimum value of the circular velocity is at the surface of the earth, when $r = r_o$. Substituting the numerical values: $g_o = 9.81$ m/sec^2 and $r_o = 6.378 \cdot 10^6$ m, we find:

$$(V_{cr})_o = 7912 \text{ m/sec}.$$

If now there were no atmosphere at the surface of the earth, then by giving a body a horizontal velocity of 7912 m/sec we should obtain an artificial satellite revolving in a surface-grazing circular orbit. This velocity is sometimes called the "first cosmic velocity". But in reality the dense strata of the atmosphere at the earth's surface will of course speedily brake the satellite's motion, so that it will lose speed and inevitably fall to the earth's surface.

With increasing height the density of the terrestrial atmosphere diminishes rather rapidly. * Thus at a height of 50 km the density amounts to only one thousandth of the density at sea level, and at a height of 100 km, to less than one millionth.

However, as is shown by the relevant calculations, even the one-millionth part of the surface density is sufficient to cause a noticeable braking action on a satellite traveling at heights of the order of 100 km.

If we imagine a satellite of spherical shape with a ratio of weight to maximum cross-sectional area equal to 200 kg/m^2, this satellite, at a height of 200 km, will be able to stay up for only about two and a half days. By this time its velocity will have become considerably less than that required for motion in a circular orbit, and the satellite will begin to descend into the less dense layers of the atmosphere and, rapidly losing speed, will fall to the earth. Therefore we may take it that artificial satellites may expediently be put up only at heights greater than 200 km. At heights of the order of 100 km the satellite would obviously have to be supplied with a very small additional forward thrust, to balance the resistance of the medium. It is proposed that satellites of this kind, having a small continuous thrust, should be called "satelloids".

To launch an artificial earth-satellite, it is necessary not only to give it the "circular" velocity V_{cr} as defined by equation (2), but also to expend some work in raising the satellite from the earth's surface to the given height h.

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* See "Priroda", 1957, No.9, pp. 3-12.

This work T_h expended in lifting may be defined as the difference between the potential energy of the body at the level of the earth's surface and its potential energy at height h.

The potential energy (in a gravitational force-field) is measured as the weight of the body in question at a stated height, multiplied by its distance r from the enter of the earth. Then at the surface of the earth the potential energy of a body of mass m will be equal to $mg_o r_o$, while at altitude $h = r - r_o$ it will be defined by the equation:

$$mgr = mg_o \frac{r_o^2}{r}$$

The difference of these two potential energies will be:

$$T_h = mg_o r_o \left(1 - \frac{r_o}{r}\right).$$

This amount of work T_h will have to be expended to raise a satellite of mass m from the surface of the earth to height h.

If we express the work T_h as a velocity V_h defined by the equation

$$T_h = \frac{mV_h^2}{2},$$

we find

$$V_h^2 = 2g_o r_o \left(1 - \frac{r_o}{r}\right).$$

Thus the total energy T_x expended in raising a satellite of mass m from the surface of the earth to height h and in there giving it the velocity V_{cr} will be equal to the sum

$$T_x = \frac{mV_h^2}{2} + \frac{mV_{cr}^2}{2}.$$

If this total work T_x is likewise expressed as a "characteristic" velocity V_x, defined in the same way as velocities V_{cr} and V_h by the expression:

$$T_x = \frac{mV_x^2}{2},$$

then, after a few elementary calculations and after substituting the numerical values of the constants g_o and r_o, we finally get for the characteristic velocity of the satellite the formula:

$$V_x = 11190 \sqrt{1 - \frac{r_o}{2r}}.$$

For $r = r_o$ the velocity V_x becomes equal to $(V_{cr})_o$ at the surface of the earth, that is, equal to 7912 m/sec, the first cosmic velocity (since in this case no work is expended in lifting the satellite). For $r \to \infty$ the velocity V_x tends toward 11,190 m/sec.

A body which acquires the latter velocity [11,190 m/sec] in the field of terrestrial gravitation will be capable of departing to an infinite distance from the earth. This velocity is sometimes called the "second cosmic velocity".

Actually, of course, if we also take into account the gravitational force of the sun and the earth's "circular" velocity in its orbit around the sun, which is equal to 29.77 km/sec, then a somewhat greater speed will be required for a body to recede to infinity from the solar system; namely, a velocity of about 16.7 km/sec in this case.

Values of V_{cr} and V_x for intermediate heights h [i.e., between 0 and ∞] are listed in Table 1 and plotted in Figure 1.

This same Table 1 gives the periods of rotation of the satellite around the earth as related to the orbital height. The higher the satellite travels above the earth's surface, the longer its orbital period. This relationship is shown graphically in Figure 2.

The characteristic velocity V_x represents the minimum total work which must be expended to elevate the satellite to an orbit of given height and then to give it the necessary circular velocity of rotation.

Thus, for instance, in order to produce a satellite rotating around the earth at a height of 200 km, we have to expend upon it, as a minimum, the amount of work defined by the characteristic velocity 8031 m/sec. Under real conditions the rocket-launching of a satellite requires some additional expenditures of energy: energy to overcome the force of air resistance in the dense strata of the atmosphere, energy to overcome the force of terrestrial gravitation during the launching period, and after that, the energy required to change the direction of the satellite's velocity.

These additional amounts of work may be figured as certain additions to the characteristic velocity V_x; they will depend on the program of velocities chosen for the satellite, and on the shape of the trajectory along which it rises to the given height.

Numerical calculations show that for actual weight relationships, accelerating forces and satellite dimensions, the total of these additional velocities may be set at about 10-15% of the characteristic velocity V_x.

Thus, allowing for real conditions of satellite firing and acceleration, the characteristic velocity V_x must be increased by 10-15% as compared with its theoretical value. If the velocity V_x was calculated to be of the order of 8 km/sec, then under real conditions it must be increased to 9 km/sec approximately.

Taking our departure from this latter value of the characteristic velocity, we are now in a position to state the power requirements for the launching of an artificial earth-satellite.

The most likely present-day apparatus capable of producing velocities of the order of 8-9 km/sec is the rocket. The velocity of a rocket, under

TABLE 1 *

Height h (km)	Circular velocity V_{cr} (m/sec)	Characteristic velocity V_x (m/sec)	Siderial period of rotation **
0	7912	7912	1 hr 24 min 25 sec
200	7791	8031	1 hr 28 min 25 sec
300	7732	8088	1 hr 30 min 27 sec
400	7675	8142	1 hr 32 min 29 sec
500	7619	8194	1 hr 34 min 32 sec
1000	7365	8431	1 hr 45 min 2 sec
2000	6903	8806	2 hrs 7 min 9 sec
4000	6203	9312	2 hrs 55 min 17 sec
6000	5679	9640	3 hrs 48 min 18 sec
∞	0	11,190	0 hrs 0 min 0 sec

* See A,M. Shternfel'd. Iskustvennyye sputniki Zemli [Artificial Earth-satellites], National Tech. Press, 1956.

** The siderial period of rotation is the time in which a heavenly body makes one complete revolution and returns to its previous position with respect to the stars.

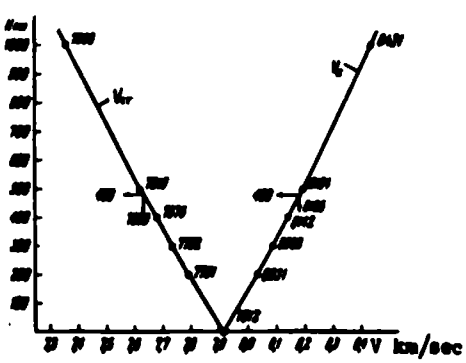

Fig. 1. Values of V_{cr} and V_x for different heights.

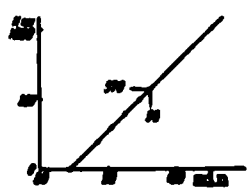

Fig. 2. Relation between period of satellite's rotation around the earth and height of its orbit.

conditions of no gravitational forces and no resistance exerted by the medium, is defined by the well-known formula of Çiolkovski: *

$$V_z = u \ln Z,$$

where V_z denotes the rocket velocity as defined by this Çiolkovski formula, u is the velocity of ejection of fuel particles from the rocket or, as it has been suggested we call it, the "effective discharge velocity", and the quantity Z, represented here by its natural logarithm, is the ratio between the initial mass of the rocket and its final mass, a ratio which is usually called "Çiolkovski's number". **

The discharge velocity u of the rocket fuel is usually determined by experimental means.

By simultaneously measuring, on the test-firing stand, the thrust P of the rocket motor and the quantity ω of fuel which it is using and discharging from the rocket each second, one can find the ratio of these quantities, the so-called specific thrust:

$$P_{sp} = \frac{P}{\omega} \left[\frac{kg}{kg/sec}\right].$$

This fundamentally important ratio indicates how many kilograms of thrust the rocket motor can supply for each kilogram of fuel expended per second.

Multiplying P_{sp} by the acceleration of gravity, $g_o = 9.81$ cm/sec^2, we find the discharge velocity u (m/sec) which has to be inserted in Çiolkovski's formula. Figure 3 is a graph of the relationship between the thrust P_{sp}, the quantity Z, and the velocity V_z.

Modern rocket motors deliver a thrust P_{sp} of the order of $250 \left[\frac{kg}{kg/sec}\right]$. Accordingly, to obtain a velocity V_z of 9 km/sec it is necessary, in the rocket, to achieve a Çiolkovski number of $Z \approx 40$.

For a single-stage rocket, this means that the initial weight, with fuel, will have to be forty times greater than the weight of the empty rocket after it has expended all its fuel. This is not yet a design possibility. The best we can achieve in a single stage rocket, with lightness of design pushed to the limit, is a ratio of $Z \approx 6$, with the useful payload usually of the order of 2% of the rocket's initial weight (at firing).

Thus in striving to achieve greater rocket velocities we must inevitably, as Çiolkovski predicted, change over to multi-stage (compound) rockets.

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* For more detail, see "Priroda" 1957, No.10, pp. 73-78.

** This Z, as also the inferior z of V_z, represents the initial of Çiolkovski's name in its German spelling (Ziolkowski). (Translator.)

In Figure 3 the vertical lines with the shaded strips on one side of them indicate the actual present-day boundaries of the regions of applicability for rockets with different numbers of stages.

For multi-stage rocket calculations, many methods exist, but the majority suffer from complexity, difficulty of visualization, involved terminology and cumbersome definitions. Recently there was published in the Journals of the British Interplanetary Society a procedure for multi-stage rocket calculations proposed by the Dutch engineer Vertregt, * a procedure which must be recognized as the most efficient. He suggested basing the computation procedure for compound multi-stage rockets on the definition of just four weights and three ratios between them, these data being sufficient for carrying out the majority of fundamental calculations for different types of step-rockets.

Figure 4 is a diagram of a step-rocket, showing the nomenclature for its principal component parts.

According to this scheme, our rocket consists of payload, stages, and sub-rockets.

The payload of a rocket may consist of instruments or passengers, including also the load-carrying structures and envelope which support and shield the instruments or passengers during flight.

A rocket stage consists of the fuel consumed by the rocket in the period of action of the said stage until the moment of its separation, plus the holders (tanks) containing this fuel, plus the motors, accessories and control apparatus, if there are such in the separating stage; also the envelope and its stiffening elements.

Sub-rocket is the name for a combination of payload and rocket stages, one of the latter being the working (operative) stage, while all the other stages, which continue in flight along with the payload of the compound rocket, constitute the "payload", as it were, of the said sub-rocket.

It is expedient to number the stages and sub-rockets in ascending order, beginning from the base of the diagram in Figure 4, and proceding toward the top.

Then for the first sub-rocket the working stage will be Stage 1, while its "payload" will be Sub-rocket 2; for Sub-rocket 2 the working stage will be Stage 2, while its "payload" will be Sub-rocket 3; for Sub-rocket 3 the working stage will be Stage 3, while the "payload" is Sub-rocket 4, and so on.

For the n^{th} sub-rocket the working stage will be the n^{th} stage, and the payload will be the weight of the /final/ useful load of the rocket.

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* See "Voprosy Raketnoi Tekhniki" [Problems of Rocket Technology], 1956, No.1, pp. 3-7.

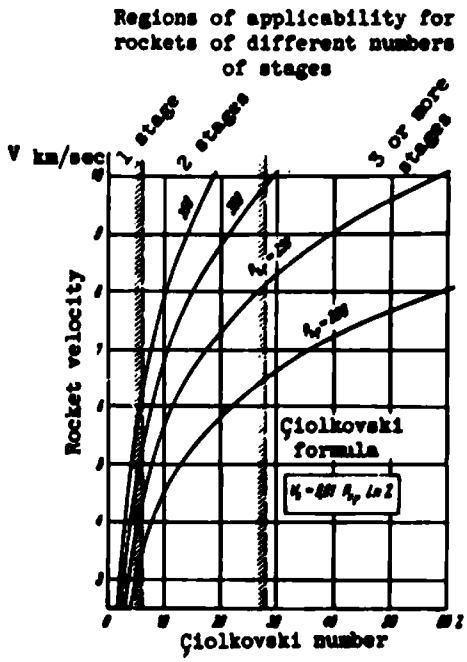

Fig. 3. Graph of relationship between specific thrust P_{sp}, mass-ratio Z and velocity V_g.

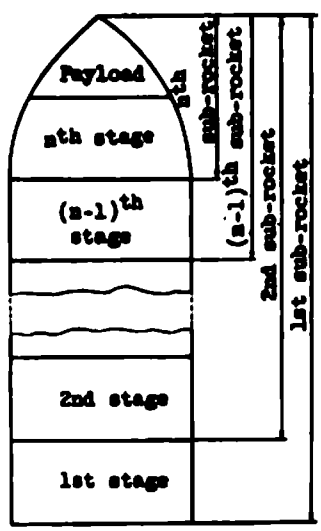

Fig. 4. Diagram of multi-stage rocket.

This manner of fixing the definitions and nomenclature of the several fundamental parts of a compound rocket is very simple, easy to remember, and makes it possible to avoid a whole series of mistakes which are always possible in the absence of such a precise systematization.

The four fundamental rocket weights which underlie the definitions will be as follows.

The weight q of the useful payload of the step-rocket. Starting from this weight, the whole rocket is planned and constructed. Consequently it is the most important weight in the calculations. As we have already said, the payload consists of instruments or living beings carried in the rocket, the stiffening structures bearing the payload, and the envelope or hull which protects it in flight.

The weight ω of the fuel expended from each stage. To this weight, which in the main is that of the fuel and oxidizer, there must be added the weight of auxiliary supplies such as, for instance, the hydrogen peroxide used to drive the turbopump unit, catalyst, compressed gas ... nitrogen, helium ... and many other chemicals expended in the given stage during its working period.

The "dry weight" Ω of the stage, that is, the total weight of the empty tanks, motors, turbopump units, valves and piping, supporting structure, envelope, control mechanisms and so forth. In other words, the weight of everything which is in the said stage and which separates along with it from the remaining parts of the rocket during flight.

The total initial weight G of the individual sub-rocket. Thus in accordance with the above-defined identifications of the various component parts of the rocket, we shall use G_1, for instance, to designate the total initial weight of the whole n-stage rocket.

ω_2 will be the total weight of fuel expended in the second stage; Ω_3 will be the "dry weight" of the third stage, and so forth.

Besides these four definitions of the weights of the most important parts of the step-rocket, three ratios between them must also be defined, as follows.

First, p will be the "relative weight" of a sub-rocket, that is, the ratio of its total initial weight to the weight of its "payload". Thus:

$$p_i = \frac{G_i}{G_{i+1}}$$

Then P, the over-all "relative weight" of the multi-stage rocket, that is, the ratio between the initial weight G_1 of an n-stage rocket and the weight q of its useful payload, will be $P = \prod_1^n p_i$, where the sign $\prod_1^n$ means the product of all the quantities p_i for all values of i from 1 to n.

The quantity P is one of the most important relationships in the multi-stage rocket.

Second, there is S, the very illuminating and important "structural factor" of the stage; it is the ratio of the total initial weight of the given stage plus fuel to its weight after expenditure of all its fuel, or in other words this "structural factor" shows the degree of perfection achieved in the structural design of this stage, in the rocket sense of the term.

Third, there is the already defined Z, the "mass ratio" of the rocket, which is called "Çiolkovski's number".

The Çiolkovski velocity for a multi-stage rocket may be found as the sum of the Çiolkovski velocities for each sub-rocket.

Then this total Çiolkovski velocity for an n-stage rocket will be defined by the expression:

$$V_z = \sum_1^n u_1 \ln z_1.$$

If for all stages the combustion-product discharge velocities $u_1, u_2 \ldots, u_n$ are to be the same and equal to u, then we have $V_z = u \ln Z$, where

$$Z = \prod_1^n z_1,$$

which is correct, of course, only under this condition that the velocities u_i for each of the n stages of the rocket are the same.

The "relative weight" of the whole n-stage rocket will be:

$$P = Z \prod_1^n \frac{S_1 - 1}{S_1 - z_1}$$

Its total weight before firing:

$$G_1 = qP.$$

The weight of the fuel in the first stage of the rocket:

$$\omega_1 = G_1 \frac{z_1 - 1}{z_1}$$

and the "dry weight" of the first-stage structure:

$$\Omega_1 = \omega_1 \frac{1}{S_1 - 1}.$$

If the "structural factor" S and the "mass ratio" Z_1 are the same for all stages and sub-rockets, then we may write still simpler equations as follows:

$$P = p^n$$

and

$$Z = s^n$$

or

$$P = Z\left(\frac{S-1}{S-Z^{-\frac{1}{n}}}\right)^n,$$

and

$$Z = P\left(\frac{S}{P^{-\frac{1}{n}}+S-1}\right)^n.$$

With these expressions, the other relationships may also be easily found.

We shall illustrate the use of the above formulae by a concrete example.

Let it be required to determine the chief characteristics of a four-stage rocket intended for accelerating an earth-satellite of weight $q = 300$ kg to a velocity $V_z = 9000$ m/sec.

This is a velocity which, as we have already stated, is capable of giving the satellite the final characteristic velocity $V_x = 8050$ m/sec which is necessary to establish it in a circular orbit at a height of a little more than 200 km above sea level and also to make up the air resistance velocity losses and gravitational losses.

For greatest simplicity of calculation, let us take the same values of u and S for each stage of this rocket, namely $u = 2400$ m/sec and $S = 4.7$. *

With $u = 2400$ m/sec, the over-all mass ratio (Çiolkovski number) for velocity $V_z = 9000$ m/sec will have to be

$$Z = 42.5$$

Then according to the equation given above, the over-all "relative weight" of the whole four-stage rocket will be:

$$P = 372.$$

Consequently the total initial weight of this four-stage rocket, according to the previously stated definition, will amount to:

$$G_1 = qP = 111.6 \text{ tons},$$

which is entirely realizable and practical, since it calls for a first-stage rocket motor with a thrust of about 220 tons, and stands for the testing of such motors are already built and in operation in the United States.

* These, according to published data, are the actual characteristics of certain types of foreign rockets, e.g., one of the American experimental rockets, the Viking.

The "dry weight" Ω_4 of the last stage, which together with the payload q will be accelerated to the final maximum velocity and will travel with the satellite in its orbit, will be given by the equation:

$$\Omega_4 = q \frac{P^{\frac{1}{S}} - 1}{S} = 216 \text{ kg.}$$

These empty tanks, closest in position to the payload of the stage, may of course be utilized as structural material to create a larger artificial earth-satellite.

The total weight of fuel in all four stages of the rocket will amount to:

$$\sum_1^4 \omega_i = G_1 \cdot \frac{S-1}{S} \cdot \frac{P-1}{P} = 87.6 \text{ tons}$$

while the "dry weight" of all its four stages will be:

$$\sum_1^4 \Omega_i = \frac{\sum_1^4 \omega_i}{S-1} = 23.7 \text{ tons.}$$

Thus a four-stage rocket, having motor and structural characteristics which are quite realizable in practice, with a weight at firing of about 112 tons and a total fuel consumption of about 88 tons, will be capable of placing a satellite in its orbit, and there communicating, to a structural mass of all-up weight about 500 kg, the velocity requisite for free rotation around the earth.

Already this is a very tangible and plausible finding, since in the above example we have utilized nothing but presently achievable characteristics (according to data published in the press *) of rockets and rocket motors operating on ordinary chemical fuel.

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* Translator's note: Before the launching of the first Soviet artificial satellite in October 1957, there was a general black-out of concrete information on the Soviet program. All Soviet writers on the subject were careful to use, for illustrative purposes, only American satellite data.